[11:00-] Z transform (Part 2)

The z-domain is a generalization of the frequency domain, and the z-transform is the generalization of the discrete-time Fourier transform. The z-transform converts convolution and difference equations into polynomials. The transfer function in the z-domain describes how an LTI system transfers input frequencies to output frequencies.

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Instead of using a formula, the inverse z-transform is more easily computed using a table of known transforms and properties.

Time domain: $x[n]$	z-domain: $X(z)$	ROC
Finite-length signal $x[n] = \sum_{k=0}^{N} x[k]\delta[n-k]$	Polynomial with powers of z $X(z) = \sum_{k=0}^{N} x[k]z^{-k}$	
Unit impulse $x[n] = \delta[n]$	Constant $X(z) = 1$	All z
Impulse delayed by n_0 samples $x[n] = \delta[n-n_0]$	z raised to the power of n_0 $X(z) = z^{-n_0}$	$z \neq 0$
Two-point impulse response $h[n] = h[0]\delta[n] + h[1]\delta[n-1]$	Polynomial in z^{-1} $H(z) = h[0] + h[1]z^{-1}$	$z \neq 0$
M-point FIR filter $\sum_{k=0}^{M} b_K \delta[n-k]$	Polynomial in z^{-1} $\sum_{k=0}^{M} b_k z^{-k}$	$z \neq 0$
Linear combination in time domain $ax_1[n] + bx_2[n]$	Linear combination in z domain $aX_1(z) + bX_2(z)$	ROC of $X_1(z)$ $\cap$ ROC of $X_1(z)$

[11:10] Relation between z domain and frequency domain

If $e^{j\hat{\omega}}$ is in the valid set of z values (region of convergence), the frequency response is

$$H\left(e^{j\hat{\omega}}\right) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

In MATLAB, the freqz command performs this substitution to compute the frequency response, but does not check if the substitution is valid.

$x[n]$ Discrete-Time Signal

Discrete-Time Fourier Transform $X(e^{j\hat{\omega}})$ $\xleftarrow{z = e^{j\hat{\omega}}}$ $X(z)$ Z-transform

only under certain cases

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega}n}$$

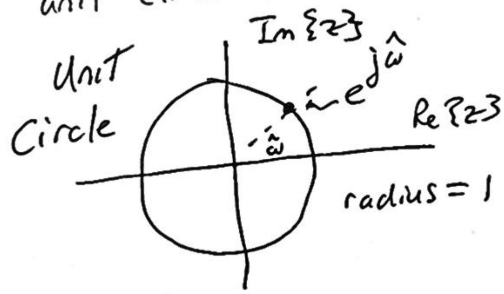
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

for a valid set of z values for which $X(z)$ is finite; and this is the Region of Convergence

$\delta[n] \overset{Z}{\longleftrightarrow} 1$ for all z

$\delta[n-1] \overset{Z}{\longleftrightarrow} z^{-1}$ for $z \neq 0$

We can only substitute $z = e^{j\hat{\omega}}$ if the region of convergence includes the unit circle.



LTI Systems

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$x[n] \longrightarrow \boxed{\text{LTI System}} \longrightarrow y[n] = x[n] * h[n]$

$= h[n] * x[n]$

$X(e^{j\hat{\omega}})$ $\quad h[n]$

$$Y(e^{j\hat{\omega}}) = X(e^{j\hat{\omega}})\, H(e^{j\hat{\omega}})$$

$$= \underbrace{H(e^{j\hat{\omega}})}_{\text{LTI system freq. response}}\, X(e^{j\hat{\omega}})$$

$X(z)$

$$Y(z) = X(z)\, H(z)$$

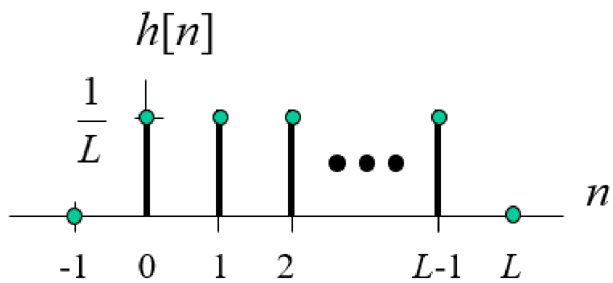
$$= H(z)\, X(z)$$

where $H(z) = \mathcal{Z}\{h[n]\}$

[11:30] Analysis of L point averaging filter

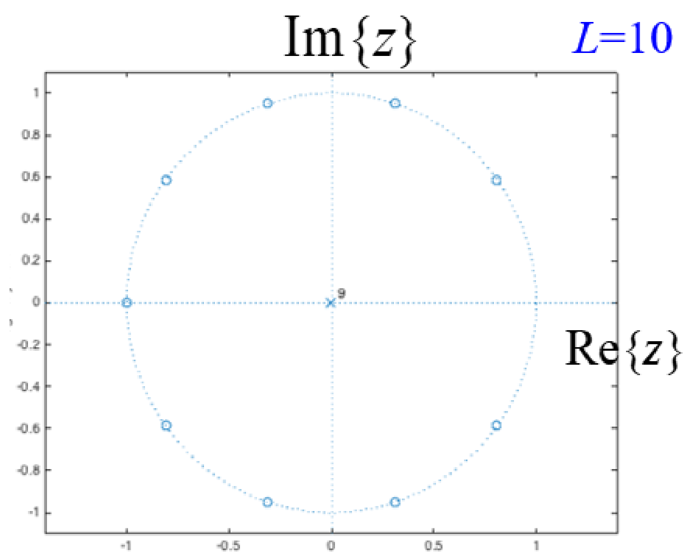
Impulse response:

$$h[n] = \frac{1}{L}\sum_{k=0}^{L-1} \delta[n-k]$$



Z-transform:

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} = \frac{1}{L}\frac{z^L - 1}{z^{L-1}(z-1)}$$



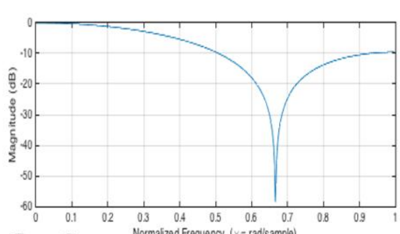
Roots of numerator (zeros):
$z = e^{j2\pi k/L}$ for $k = 1,2, \ldots L-1$

Roots of denominator (poles):
$z = 0$ repeated $L-1$ times

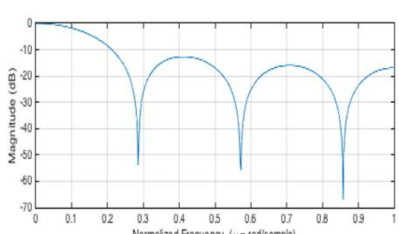
The zero at $z = 1$ cancels the pole at $z = 1$.

To determine the type of frequency selectivity (lowpass, bandpass, highpass, etc), we examine the magnitude response:

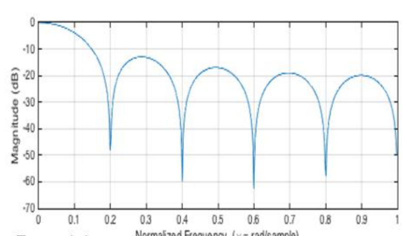
$$\left|H\left(e^{j\hat{\omega}}\right)\right| = \frac{1}{L}\left|e^{j\hat{\omega}} - e^{j2\pi k/L}\right|$$



$L = 3$

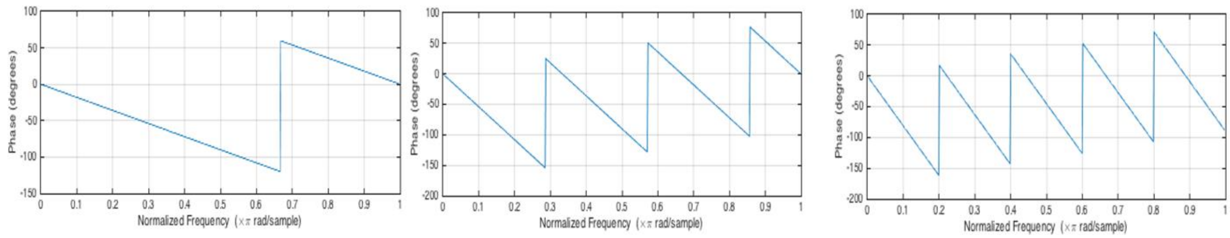


$L = 7$



$L = 11$

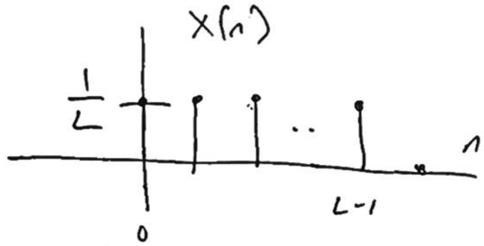
The phase response $\angle H(e^{j\hat{\omega}})$ is linear except at $\hat{\omega} = 2\pi k/L$. However. The magnitude response at $\hat{\omega} = 2\pi k/L$ is zero, so the phase is linear for all frequencies that make it through the filter.



$$\text{Group delay }(\hat{\omega}) = -\frac{d}{d\hat{\omega}}\angle H(\hat{\omega}) = \frac{L-1}{2}\text{ samples}$$

Thus, the group delay does not vary as a function of frequency.

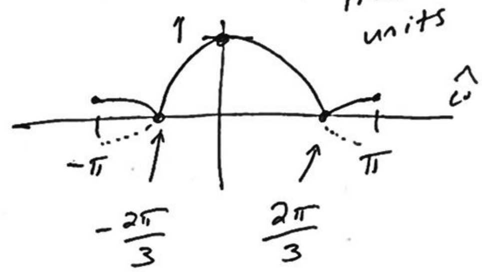
Averaging Filter



$L = 3$

Lowpass Filter

$|X(e^{j\hat{\omega}})|$ in linear units



$L = 3$

Phase Resp. $\measuredangle H(\hat{\omega}) = -\hat{\omega}$

$= -\frac{L-1}{2}\hat{\omega}$

if we exclude the frequencies zeroed out by the magnitude resp.

Mag. Rep. Frequencies at $\pm\frac{2\pi}{3}$ are zeroed out by the three-point averaging filter

Definition

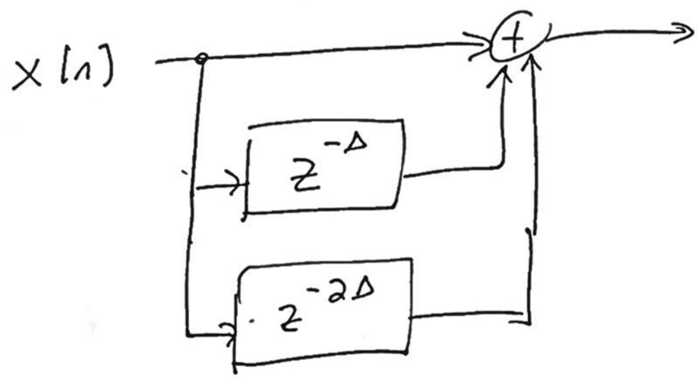
for the averaging filter

$$\text{Group Delay}(\hat{\omega}) = -\frac{d}{d\hat{\omega}}\measuredangle H(\hat{\omega}) = \frac{L-1}{2}\text{ samples}$$

all frequencies in the input experience the ~~same~~ same delay through the averaging filter

[12:00] Demo: Audio reverb and delay

Audio Controlled Delay



Long delay of 1 second.
Sampling is f_s samples/second: $\Delta = f_s \cdot t_0$ will give a delay of t_0 seconds

Short delay effects.
~10-40 ms.
add a guitar playing with adding two short-delay versions to make it sound three guitars are playing.

Chorusing